

An Analytical Solution for the Rotational Dynamics of Axisymmetric Rigid Bodies Under Time-Varying Body-Fixed Torques

Journal Title
XX(X):1–11
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sagepub.co.uk/journalsPermissions.nav
DOI: 10.1177/ToBeAssigned
www.sagepub.com/

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Abstract

This article presents a general analytical solution concerning the attitude motion of axisymmetric rigid bodies subject to time-varying body-fixed torques. Leveraging the axisymmetry feature of the body, it is demonstrated that the system becomes linear and time-variant, allowing the use of a state space approach. This leads to two key contributions: (i) A new closed-form analytical solution is derived by applying a single series expansion to the system's zero-state response. Central to this approach is a novel recursive formulation that streamlines the computation of high-order derivatives, enabling a unified global approximation of system dynamics. This eliminates the need for separate expansions of each torque component and supports efficient offline construction of high-precision analytical representations. (ii) The closed-form structure enables analytical gradients, making the approach highly suitable for embedded optimal control and embedded machine learning applications under constrained computational resources. Simulation results demonstrate both the accuracy and computational effectiveness of the proposed solution.

Keywords

Introduction

The general attitude motion of a rigid body is described by Euler's equations¹. They constitute a set of three coupled differential equations, and there is no known general analytical solution to them. As a result, numerical solutions are often employed to describe the attitude behavior and control characteristics of a given rigid body.

While numerical solutions are proven to be very useful, their accuracy may be enhanced with the expense of increased computational load and complexity². Analytical models, on the other hand, may provide insights in understanding the qualitative behavior of a dynamic system while necessitating relatively less computational resources^{3,4}. The researchers demonstrated that even basic heuristic analytical results could be used for maneuver analysis of aerial vehicles^{5–7}.

The low computational complexity inherent to analytical solutions makes them well-suited for closed-loop navigation control of autonomous aerial vehicles^{8–11}. In particular, the growing demand for cost-effective quadrotor platforms necessitates the use of analytical expressions to model attitude dynamics within the limited capabilities of onboard computing systems¹². Moreover, recent developments in embedded MPC (Model Predictive Control) and onboard machine learning stand to benefit from such expressions, as they facilitate aggressive maneuvering and efficient trajectory planning^{13,14}. Analytical formulations are also valuable in solving two-point boundary value problems, particularly as effective initial guesses in shooting method algorithms⁵. Taken together, these observations underscore the relevance of analytical solutions in the analysis and control of aerial vehicle attitude dynamics. Motivated by this need, this paper proposes a new analytical solution to Euler's

equation for the case of axisymmetric rigid bodies, subject to time-varying body-fixed torques.

The paper is organized as follows. In the next subsections, a brief literature review is provided and the contributions of the article is summarized. The proposed analytical solution is explained in the second section in detail. Results are presented and discussed in the third section. The article is concluded in the fourth section.

Literature

The general solution of Euler's equation attracted many researchers throughout the history. For the case of zero body-fixed torques, Poinot constructed a geometric visualization method named after him to describe the attitude motion of a rigid body^{15,16}. Routh also tackled the same problem and analytically solved Euler's equation for the case of zero body-fixed torques¹⁷. Recent attempts to revisit Routh's study have provided computer-aided visualization and open-source coding, demonstrating the presence of bifurcations in the solution^{7,18–20}.

To tackle the cases with non-zero body-fixed torques, Longuski proposed a solution in which the rigid body is assumed to be near symmetric and body-fixed torques are constant²¹. Later, this solution is expressed using more compact expressions⁶, and generalized for the case when the analytical solution of the axial angular velocity is given a

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priori²². A variety of other solutions for axially symmetric and near symmetric bodies may be seen in the literature^{23–28}.

Concerning the attitude motion of asymmetric rigid bodies, Longuski and Tsiotras and proposed an analytical solution in which body-fixed torques are constant²⁹. Further studies in this matter examined a multitude of different cases, e.g., the body is subjected to a time-varying torque acting along the largest principal axis, where the torque magnitude is proportional to the squared norm of the angular velocity plus a constant, while the remaining body-fixed torque components are identically zero³⁰, the body is subject to constant torque along one of the principle axes³¹, the body is subject to zero axial torque³², the body is subject to torque that is aligned with the momentum vector³³. While these studies are useful in their own domain, they may not be directly applicable when the body-fixed torques are time-varying.

In the past decade, a related body of work has examined the rotational dynamics of rigid bodies augmented with gyrostatic moments that may arise from internal rotors. In these studies, the governing equations are modified due to the presence of gyrostatic moments, while the externally applied body-fixed torques are typically assumed to be constant. Representative examples include analyses of charged or gyrostatically actuated rigid bodies under constant torques^{34,35}, investigations of symmetric and asymmetric rigid bodies subjected to constant or time-dependent gyrostatic torques^{36–38}, rigid body rotation around a fixed point, i.e., Euler-Poisson equations^{39–43}, motion of rotating rigid bodies with energy dissipation^{44–47}, and studies addressing the spatial motion and stationary solutions of gyrostic systems using both analytical and computer-algebra-based techniques⁴⁸. These contributions provide valuable insight into the influence of internal angular momentum on rigid-body dynamics; however, their assumptions of constant body-fixed torques distinguish them from the present work, which focuses on externally applied, time-dependent body-fixed torques within the absence of internal rotors.

In the literature, only a limited number of studies have addressed the case of time-varying body-fixed torques. Longuski and Tsiotras investigated the rotational motion of near-symmetric rigid bodies without internal rotors, allowing the transverse body-fixed torques to be time-varying while assuming a constant axial torque. In a similar manner, Amer et al. derived a solution for the motion of a charged rigid body subjected to time-varying body-fixed torques with a constant axial torque component⁴⁹. Distinguishing from these works, Tsiotras and Longuski developed analytical solutions for rigid-body dynamics with time-varying body-fixed torques acting along all three principal axes⁵⁰. The present work follows this line of inquiry by considering an axisymmetric rigid body without internal rotors, subjected to time-varying body-fixed moments about all principal axes.

In their formulation, Tsiotras and Longuski represented each torque component using an independent power-series expansion. Although this approach accommodates a wide range of practical scenarios, the use of separate series for each moment introduces multiple sources of approximation, potentially leading to increased cumulative error. This observation motivates the present study, which seeks to

reduce approximation error by employing a single expansion applied to the overall zero-state solution.

Contribution

Despite the presence of useful analytical derivations, there are still aspects that require further investigation for the case of time-varying body-fixed torques. In response to this gap, this article offers two contributions:

i) A new closed-form analytical solution is presented for the zero-state response of Euler's equations under time-varying body-fixed torques. This is made possible by a novel recursive formulation that efficiently computes high-order derivatives, circumventing the symbolic intractability of traditional approaches. The result is a compact expression with a unified global approximation error, supporting offline construction of high-fidelity solutions and accommodating flexible torque parameterizations such as Gaussian kernels, Fourier series, or power series.

ii) The closed-form structure enables analytical gradients, making the method particularly well-suited for embedded optimal control and embedded machine learning applications under constrained computational resources. Compared to numerical solvers, the recursive nature of the solution significantly reduces online computational demands, offering a practical and efficient alternative for low-cost platforms such as quadrotors with limited onboard processing capability.

Rotational Dynamics of Axisymmetric Bodies

For a given rigid body, Euler's equation for principal axes describes its attitude^{1,17}:

$$M_x = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z \quad (1)$$

$$M_y = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x \quad (2)$$

$$M_z = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \quad (3)$$

where, $\mathbf{M} = [M_x, M_y, M_z]^\top$ denote the body-fixed torques, I_{xx}, I_{yy}, I_{zz} are moment of inertia values about the principal axes, $\boldsymbol{\omega} = [\omega_x, \omega_y, \omega_z]^\top$ stand for the angular velocity. Compared to diagonal elements, the products of inertia are deemed to be negligible.

Although the eqs. (1)-(3) represent the attitude motion of rigid bodies in general, no explicit solution of this system equation exists for the case of time-varying body-fixed torques⁵. Yet, the second term in (3) is zero for axisymmetric bodies ($I_{xx} = I_{yy}$) and negligible for near symmetric bodies ($I_{xx} \approx I_{yy}$). In this sense, a common diagonal inertia term for x and y axes is considered: ($I_{xx} = I_{yy} = I_{aa}$)

With this in mind, Euler's equations take the following form:

$$M_x = I_{aa}\dot{\omega}_x - (I_{aa} - I_{zz})\omega_y\omega_z \quad (4)$$

$$M_y = I_{aa}\dot{\omega}_y - (I_{zz} - I_{aa})\omega_z\omega_x \quad (5)$$

$$M_z = I_{zz}\dot{\omega}_z \quad (6)$$

Solving Euler's Equations for Axisymmetry

As may be observed, the eq. (6) is decoupled and it can be solved for ω_z :

$$\omega_z = \omega_{z0} + \frac{1}{I_{zz}} \int_{t_0}^t M_z dt \quad (7)$$

In (7), ω_{z0} denotes initial angular velocity about the z-axis. The parameters t and t_0 are time variable and initial time. Plugging (7) to (4) and (5), the following equation set is yielded:

$$\dot{\omega}_y = \frac{M_y}{I_{aa}} + \frac{(I_{zz} - I_{aa})}{I_{aa}} \omega_x \left(\omega_{z0} + \frac{1}{I_{zz}} \int_{t_0}^t M_z dt \right) \quad (8)$$

$$\dot{\omega}_x = \frac{M_x}{I_{aa}} + \frac{(I_{aa} - I_{zz})}{I_{aa}} \omega_y \left(\omega_{z0} + \frac{1}{I_{zz}} \int_{t_0}^t M_z dt \right) \quad (9)$$

Assigning $x_1 = \omega_y$ and $x_2 = \omega_x$, (8) and (9) can be expressed in the state-space form:

$$\dot{x} = Ax + Bu \quad (10)$$

where $x = [x_1 \ x_2]^T$, and $u = [M_y \ M_x]^T$. The matrices A and B are as follows:

$$A = \begin{bmatrix} 0 & -\eta \\ \eta & 0 \end{bmatrix}, \quad B = \frac{1}{I_{aa}} \mathbb{1}_{2 \times 2} \quad (11)$$

where

$$\eta = \kappa_a I_{zz} \omega_{z0} + \kappa_a \int_{t_0}^t M_z dt \quad (12)$$

$$\kappa_a = \frac{I_{aa} - I_{zz}}{I_{aa} I_{zz}} \quad (13)$$

The analytical solution of (10) exists as $A(t)$ and $\int_{t_0}^t A(\tau) d\tau$ commute for all t such that⁵¹

$$A(t) \left(\int_{t_0}^t A(\tau) d\tau \right) = \left(\int_{t_0}^t A(\tau) d\tau \right) A(t). \quad (14)$$

In (14), τ is a dummy variable. Indeed, the matrix $A(t)$ expressed in (11) holds this property. Therefore, the total solution is the superposition of zero input solution $x_{zi}(t)$, i.e., $[M_y(t) \ M_x(t)]^T = [0 \ 0]^T$ and zero state solution, i.e., $[\omega_{y0} \ \omega_{x0}]^T = [0 \ 0]^T$:

$$x(t) = \overbrace{\Phi(t, t_0) x(t_0)}^{x_{zi}(t)} + \overbrace{\int_{t_0}^t \Phi(t, \tau) B(\tau) u(\tau) d\tau}^{x_{zs}(t)}, \quad (15)$$

In (15), $\Phi(t, t_0)$ is the state transition matrix, and $x(t_0)$ is the initial condition vector such that $x(t_0) = [\omega_{y0} \ \omega_{x0}]^T$. $u(\tau)$ stands for the time-dependent input vector as $u(\tau) = [M_y(\tau) \ M_x(\tau)]^T$. The state transition matrix holds the following property:

$$\frac{d}{dt} \Phi(t, t_0) = A(t) \Phi(t, t_0) \quad (16)$$

As the commutation property given in (14) holds true, the state transition matrix can be computed via the following matrix exponential:

$$\Phi(t, t_0) = e^{\int_{t_0}^t A(\tau) d\tau} \quad (17)$$

The Zero Input Solution ($x_{zi}(t)$): Inserting (11) and (12) into (17), the state transition matrix can be obtained as in the following.

$$\Phi(t, t_0) = \begin{bmatrix} \cos \alpha_s & -\sin \alpha_s \\ \sin \alpha_s & \cos \alpha_s \end{bmatrix} \quad (18)$$

In (18), α_s appears to be the double integration of z-axis moment and is dimensionless. This quantity is yielded through the matrix exponential given in (17):

$$\alpha_s = \kappa_a \left(\int_{t_0}^t \int_{t_0}^{\tau} M_z(\tau) d\tau d\tau + I_{zz} \omega_{z0} (t - t_0) \right) \quad (19)$$

Using (18)-(19), the zero input solution is obtained by computing $x_{zi}(t) = \Phi(t, t_0) x(t_0)$:

$$x_{zi}(t) = \begin{bmatrix} \omega_{y0} \cos \alpha_s - \omega_{x0} \sin \alpha_s \\ \omega_{x0} \cos \alpha_s + \omega_{y0} \sin \alpha_s \end{bmatrix} \quad (20)$$

If the body-fixed axial moment function is twice integrable, then α_s can be obtained from (19), and substituting it into (20) yields the zero-input solution. Thus, this integrability condition is the only requirement in this case.

The Formal Zero State Solution ($x_{zs}(t)$): The zero state solution can be expressed as indicated in (15):

$$x_{zs}(t) = \int_{t_0}^t \Phi(t, \tau) B u(\tau) d\tau \quad (21)$$

where the state transition matrix appears to be a rotation matrix:

$$\Phi(t, \tau) = \begin{bmatrix} \cos \beta_s & -\sin \beta_s \\ \sin \beta_s & \cos \beta_s \end{bmatrix} \quad (22)$$

In (24), the dimensionless parameter β_s is expressed as below:

$$\beta_s = \kappa_a \left(\int_{\tau}^t \int_{t_0}^{\tau} M_z(\tau) d\tau d\tau + I_{zz} \omega_{z0} (t - \tau) \right) \quad (23)$$

Note that for $\tau = t$, $\beta_s(t, t) = 0$ and therefore $\Phi(t, t) = \mathbb{1}_{2 \times 2}$. Combining (21), (22), and (23), the formal zero state solution is obtained as in the following:

$$x_{zs}(t) = \begin{bmatrix} \int_{t_0}^t \left(\frac{M_y(\tau)}{I_{aa}} \cos \beta_s - \frac{M_x(\tau)}{I_{aa}} \sin \beta_s \right) d\tau \\ \int_{t_0}^t \left(\frac{M_x(\tau)}{I_{aa}} \cos \beta_s + \frac{M_y(\tau)}{I_{aa}} \sin \beta_s \right) d\tau \end{bmatrix} \quad (24)$$

In this case, even if the axial moment is twice integrable, β_s appears within the trigonometric functions $\cos$ and $\sin$ with the resulting expression undergoing an additional integration. This explains the emergence of Fresnel integrals for the case of constant axial moment²¹. Indeed, a time-varying axial moment would further complicate the solution.

The Workable Zero State Solution ($x_{zs}^*(t)$): Since a closed-form analytical solution for $x_{zs}(t)$ is not possible, one can perform Taylor's expansion to (21) around $t = t_0$ to obtain a workable closed-form solution, namely $x_{zs}^*(t)$. To this end, the eq. (21) must be continuously differentiated with respect to time, up to the desired level of convergence. The first time derivative is yielded as follows, by respecting Leibniz's rule:

$$\begin{aligned} \frac{dx_{zs}(t)}{dt} &= \Phi(t, t_0) B u(t) \\ &+ \int_{t_0}^t \frac{\partial}{\partial t} \Phi(t, \tau) B u(\tau) d\tau \end{aligned} \quad (25)$$

Recall that the matrix B is constant; see the eq. (11). Since the state transition matrix $\Phi(t, \tau)$ is a rotation matrix, its first time derivative can be computed as follows:

$$\frac{\partial \Phi(t, \tau)}{\partial t} = \frac{\partial \beta_s}{\partial t} S \Phi(t, \tau) \quad (26)$$

where S is a skew-symmetric matrix:

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (27)$$

Furthermore, the first time derivative of β_s only depends on t and does not depend on τ :

$$\frac{\partial \beta_s}{\partial t} = \kappa_a \left(I_{zz} \omega_{z0} + \int_{t_0}^t M_z(t) dt \right) \quad (28)$$

In light of these results, (25) can be expressed as in the following:

$$\begin{aligned} \frac{dx_{zs}(t)}{dt} &= B u(t) + \frac{\partial \beta_s}{\partial t} S \int_{t_0}^t \Phi(t, \tau) B u(\tau) d\tau \\ &= B u(t) + \frac{\partial \beta_s}{\partial t} S x_{zs}(t) \end{aligned} \quad (29)$$

Since (29) is a recursive equation, the continuous time derivatives of $x_{zs}(t)$ follow the binomial expansion rule and can be generalized as below:

$$\begin{aligned} \frac{d^{n+1} x_{zs}(t)}{dt^{n+1}} &= \sum_{k=0}^n \left(C_{\{n,k\}} \frac{\partial^{n-k+1} \beta_s}{\partial t^{n-k+1}} S \frac{d^k x_{zs}(t)}{dt^k} \right) \\ &+ B \frac{d^{n+1} u(t)}{dt^{n+1}} \end{aligned} \quad (30)$$

In (30), $C_{\{n,k\}}$ are binomial expansion coefficients; $C_{\{n,k\}} = \frac{n!}{k!(n-k)!}$; $n, k \in \mathbb{Z}$. The first five time derivatives are derived manually and reported in the Appendix for reference. In addition, time derivatives of β_s can be computed as follows for $m \geq 2$, $m \in \mathbb{Z}$:

$$\frac{\partial^m \beta_s}{\partial t^m} = \kappa_a \frac{\partial^{m-2} M_z(t)}{\partial t^{m-2}} \quad (31)$$

Note that the eq. (31) can be used when 2^{nd} and higher time derivatives of β_s is required. For its 1^{st} time derivative,

one can use (28). The equations (28), (30), and (31) enable the recursive computation of all continuous time derivatives of $x_{zs}(t)$, facilitating the construction of Taylor series expansions to obtain $x_{zs}^*(t)$ to the desired level of accuracy:

$$\begin{aligned} x_{zs}^*(t) &= \sum_{n=0}^{n_{max}} \frac{1}{n!} \frac{d^n x_{zs}}{dt^n} \Big|_{t=t_0} (t - t_0)^n \\ &+ \mathcal{O}((t - t_0)^{n_{max}+1}) \end{aligned} \quad (32)$$

where n_{max} represents the highest order of the time derivative included in the Taylor series expansion, and the term $\mathcal{O}((t - t_0)^{n_{max}+1})$ stands for the asymptotic order of the truncation error.

Formal Total Solution and Its Practical Approximation: An important property of the system described by (10) is its linearity. Owing to this feature, the total system response can be expressed as the superposition of the zero-input and zero-state solutions, as given by (15). The decomposition, namely $x(t) = x_{zi}(t) + x_{zs}(t)$, yields the formal total solution as follows:

$$\begin{aligned} x(t) &= \begin{bmatrix} \omega_{y0} \cos \alpha_s - \omega_{x0} \sin \alpha_s \\ \omega_{x0} \cos \alpha_s + \omega_{y0} \sin \alpha_s \end{bmatrix} \\ &+ \begin{bmatrix} \int_{t_0}^t \left(\frac{M_y(\tau)}{I_{aa}} \cos \beta_s - \frac{M_x(\tau)}{I_{aa}} \sin \beta_s \right) d\tau \\ \int_{t_0}^t \left(\frac{M_x(\tau)}{I_{aa}} \cos \beta_s + \frac{M_y(\tau)}{I_{aa}} \sin \beta_s \right) d\tau \end{bmatrix} \end{aligned} \quad (33)$$

While the zero-input component $x_{zi}(t)$ admits a closed-form expression under mild regularity assumptions on $M_z(t)$, the zero-state component $x_{zs}(t)$ involves time integrals that, for general time-varying body-fixed torques, do not admit an analytical closed-form solution. Consequently, (33) should be interpreted as the theoretical or formal solution of the system.

To obtain a practically computable expression, the zero-state solution was approximated via a Taylor series expansion; see previous subsection. This expansion replaces the intractable integral terms in $x_{zs}(t)$ with a finite-order polynomial whose coefficients can be computed recursively, as described previously. Using this approximation the final workable closed form solution can be yielded through the decomposition $x^*(t) = x_{zi}(t) + x_{zs}^*(t)$ as below:

$$\begin{aligned} x^*(t) &= \begin{bmatrix} \omega_{y0} \cos \alpha_s - \omega_{x0} \sin \alpha_s \\ \omega_{x0} \cos \alpha_s + \omega_{y0} \sin \alpha_s \end{bmatrix} \\ &+ \sum_{n=0}^{n_{max}} \frac{1}{n!} \frac{d^n x_{zs}}{dt^n} \Big|_{t=t_0} (t - t_0)^n \\ &+ \mathcal{O}((t - t_0)^{n_{max}+1}) \end{aligned} \quad (34)$$

Results and Discussion

In order to demonstrate the viability of the proposed solutions, a series of case studies were conducted.

Table 1. Simulation #1 Parameters

Parameter	Value
Inertia (kgm^2)	$\text{diag}\{8810.8, 8810.8, 4721.8\}$
$\omega(t_0)$ (rad/s)	$[0.42 \ 0.63 \ -1.75]^\top$
Period (s)	$t_0 = 3$ and $t_{\text{end}} = 123$
Sampling (s)	0.005

Table 2. Simulation #2: Parameters

Parameter	Value
Inertia (kgm^2)	$\text{diag}\{0.0291, 0.0291, 0.0552\}$
$\omega(t_0)$ (rad/s)	$[0.42 \ 0.63 \ -0.75]^\top$
Period (s)	$t_0 = 0$ and $t_{\text{end}} = 10.5$
Sampling (s)	0.01

Case Study #1: Confirmation of Formal Solution

To verify the proposed method's formal solution before assessing the final and workable closed-form solution, the analytical solution in eq. (33) was primarily considered. To this end, a numerical simulation was conducted using MATLAB R2024b⁵², on a Windows 11 PC with an Intel(R) Core(TM) i7-1370U CPU (1.90 GHz) and 64 GB RAM. The trapezoidal rule was employed concerning the integrations in (33) when necessary. For verification, Euler's equations (eqs. (1)-(3)) were also numerically solved using MATLAB's ode113 function. In this simulation, the inertia values of the Cassini spacecraft were used⁵³; however, for proof of concept, I_{yy} was equalized to I_{xx} . Simulation parameters are tabulated in Table 1. As for time-varying body-fixed torques, the following arbitrary functions were used:

$$M_x = 300(1 - \arctan(0.1\pi t)) \quad (35)$$

$$M_y = 3e^{-0.0005t}(80 \cos(2\pi t) - 20) \quad (36)$$

$$M_z = 300 \sin(0.2\pi t) + 15(1 - e^{-0.025t}) \quad (37)$$

There is no deliberate choice in these equations and one can assign any type of function. The simulation result is depicted in Fig. 1, where solid blue and dashed red lines indicate the proposed analytical and numerical solutions, respectively. As observed in this figure, the difference between these solutions is negligible and no drift was observed. In order to test the algorithm for different time-varying torque functions, refer to the Code Ocean repository in⁵⁴.

Case Study #2: Workable Closed-Form Analytical Solution

To verify the feasibility of the proposed closed-form solution outlined in (34), a simulation was conducted using the inertia properties of a 3DR IRIS* quadrotor⁵⁵. The simulation parameters are listed in Table 2. Owing to its geometry, the quadrotor is naturally axisymmetric.

To demonstrate the computational efficiency of the proposed approach, the simulation was executed on a low-cost Arduino Uno module equipped with an 8-bit ATmega328P microcontroller operating at 16 MHz and featuring only 2 KB of SRAM. As a baseline for comparison, Heun's method was used, since it offers a reasonable trade-off between computational cost and

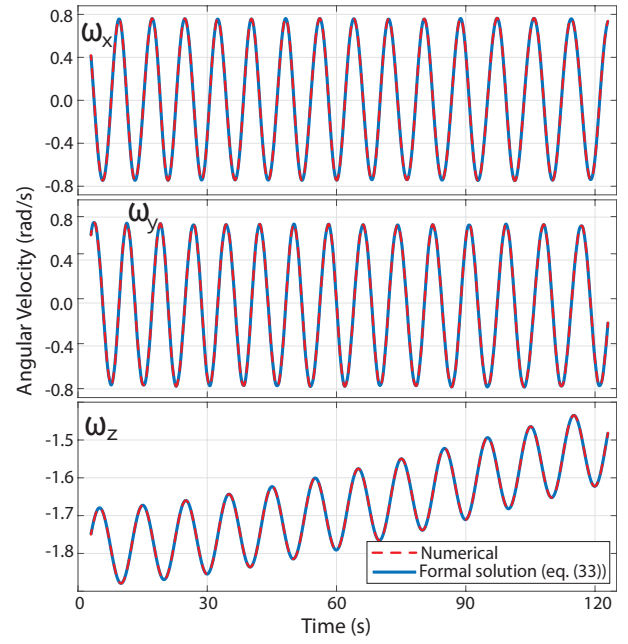


Figure 1. Case Study #1: The formal confirmed with a numerical solution. Since the integrals in (33) cannot be evaluated analytically, a simple trapezoidal method was employed.

accuracy⁵⁶. For an extended representational capability, the Taylor series were truncated after the 23rd term in the proposed method. This cutoff was determined empirically by incrementally increasing the expansion order until no significant improvement in accuracy was observed. To represent a diverse set of body-fixed moment profiles, various kernel functions were used, including Gaussian, Fourier-based forms, and a low-slope ramp to simulate drift, demonstrating the flexibility and broad applicability of the proposed method.

$$M_x = 7e^{-0.0386(t-5)^2} - 1 \quad (38)$$

$$M_y = -2 \sin(1.0472 - t) - 0.4 - 4 \sin(0.1745 - 0.5t) \quad (39)$$

$$M_z = 10^{-4}t \quad (40)$$

The results are displayed in Figs. 2–4. In Fig. 2, angular velocities obtained via Heun's numerical method (dashed red) and the proposed method (solid blue) are displayed. As observed, the discrepancy between the solutions is negligible for the first 9.5 seconds. After this instant, the proposed closed-form solution begins to diverge, as the representation capability of the Taylor series deteriorates.

The approximation error caused by the truncation of series expansion depends on the level of truncation as well as the temporal variation of the body-fixed torque profiles. While increasing the truncation order monotonically extends the validity horizon of the approximation, it does not eliminate divergence, reflecting the local-in-time nature of Taylor-based solutions under general time-varying excitations.

* <https://www.arducopter.co.uk/iris-quadcopter-uav.html>

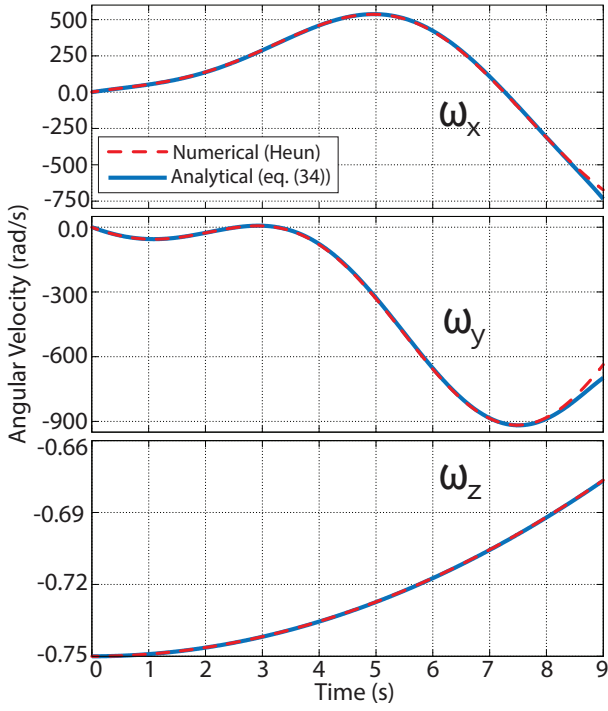


Figure 2. Case Study #2: Closed-form solution; see eq. (34). The proposed analytical solution is validated on an Arduino Uno module by comparing it against Heun's numerical solver.

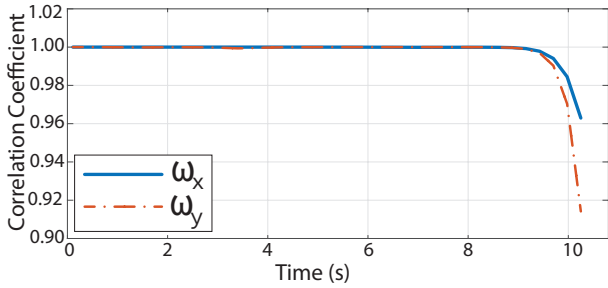


Figure 3. Case Study #2: Closed-form solution; see eq. (34). The agreement between the proposed analytical solution and the ground truth is demonstrated via Pearson's correlation coefficient, computed over a sliding time window.

Therefore, it may not be possible to express a global approximation error; however, it is possible to analyze it case by case. To this end, the Pearson correlation coefficients were computed over a sliding time window where the temporal agreement between the analytical and numerical solutions is displayed; see Fig. 3. The correlation was calculated incrementally from $t = 0$ to increasing final times (e.g., $t = 1$, $t = 2$), revealing where agreement begins to degrade.

To further investigate computational considerations, the execution time and computational throughput are presented in Fig. 4. Over a 10.5 s simulation period, the numerical method completed in 1514.9 ms, while the proposed method required only 78.43 ms, yielding a speed-up factor of 19.3. Similarly, the proposed method achieved a computational throughput of 13388 steps/sec, significantly outperforming Heun's method, which reached only 693 steps/sec. By computing the Taylor expansion coefficients of the solution offline using a recursive formulation, the proposed approach

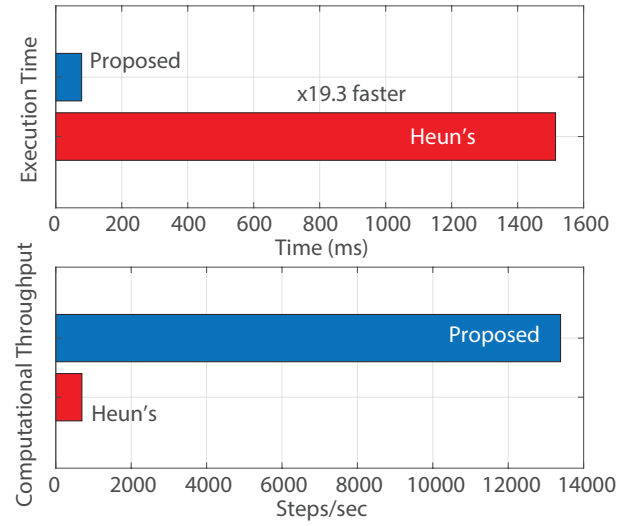


Figure 4. Case Study #2: Closed-form solution; see eq. (34). The computational load of the proposed analytical solution is evaluated and compared with respect to Heun's method.

reduces the online computation to a simple polynomial evaluation via Horner's method, thereby minimizing the computational burden on onboard microcontrollers.

Case Study #3: Optimal Control via Analytical Gradients

Another practical advantage of the proposed analytical solution lies in the availability of closed-form analytical gradients, which offer significant potential for embedded optimal control and embedded machine learning implementations. The objective here is not to introduce a new velocity controller, but rather to demonstrate how these analytically derived gradients facilitate efficient optimization, even on severely resource-constrained hardware. To illustrate this, the inertia properties of a 3DR IRIS quadrotor were considered, and an optimal control routine was implemented on a low-cost Arduino Uno platform featuring an 8-bit ATmega328P microcontroller operating at 16 MHz with 2 KB of SRAM.

The control torques, M_x and M_y were represented as weighted sums of Gaussian kernel functions as follows, where each kernel was centered at a fixed location c_i and scaled by a learnable weight $k_{x,i}$ or $k_{y,i}$.

$$M_x = \sum_{i=1}^N k_{x,i} e^{-\frac{(t-c_i)^2}{2\sigma^2}} \quad (41)$$

$$M_y = \sum_{i=1}^N k_{y,i} e^{-\frac{(t-c_i)^2}{2\sigma^2}} \quad (42)$$

The kernel width σ was held constant across all terms. The parameter N was set to 10. This structure allowed smooth and low-dimensional control signal synthesis. The kernel centers were distributed uniformly over the time interval, and the optimization was performed only over the kernel weights. For simplicity, M_z is kept the same as in (40).

To demonstrate the utility of the proposed approach, a standard angular velocity regulation task was considered, in

which both ω_x and ω_y were driven to zero from nonzero initial conditions. This task is representative of damping or stabilization behaviors commonly required in aerial vehicle control. In this context, angular velocity references ($\omega_x^{ref}, \omega_y^{ref}$) were defined as follows:

$$\omega_x^{ref} = \omega_{x0} e^{-10t} \quad (43)$$

$$\omega_y^{ref} = \omega_{y0} e^{-10t} \quad (44)$$

The loss function was formulated as the squared Euclidean norm of the error, representing a standard quadratic cost.

$$\mathcal{L} = -p_1 (e_x)^2 - p_2 (e_y)^2 \quad (45)$$

where $e_x = \omega_x - \omega_x^{ref}$ and $e_y = \omega_y - \omega_y^{ref}$. Moreover, p_1 and p_2 are the weighting coefficients.

The analytical gradient of the loss function with respect to learnable parameters $k_{x,i}$ and $k_{y,i}$ can be derived as in the following.

$$\frac{\partial \mathcal{L}}{\partial k} = -2 \left(e_x \frac{\partial \omega_x}{\partial k} p_1 + e_y \frac{\partial \omega_y}{\partial k} p_2 \right) \quad (46)$$

in which $k = [k_{x,1}, k_{x,2}, \dots, k_{x,10}, k_{y,1}, k_{y,2}, \dots, k_{y,10}]^T$. Given the proposed analytical solution, the gradients $\frac{\partial \omega_x}{\partial k}$ and $\frac{\partial \omega_y}{\partial k}$ can be computed in closed form, prior to optimization.

Concerning the optimal angular velocity regularization, the following algorithm was implemented.

Algorithm 1 Optimal angular velocity regularization with analytical gradients

- 1: **Input:** Learning rate η , kernel centers c_i , kernel width σ , maximum iterations I_{max} , loss tolerance ϵ , weights p_1, p_2
- 2: **Initialize:** Random kernel weights $k_{x,i}$ and $k_{y,i}$ for $i = 1, \dots, N$
- 3: **for** each time step t **do**
- 4: $i \leftarrow 0$
- 5: **while** $\mathcal{L} > \epsilon$ and $i < I_{max}$ **do**
- 6: Compute M_x and M_y using Gaussian kernels (see (41)–(42))
- 7: Apply the moments to obtain ω_x and ω_y
- 8: Compute the loss $\mathcal{L}$ (see (45))
- 9: Compute the analytical gradients (see (46))
- 10: Update the weights: $k \leftarrow k + \eta \frac{\partial \mathcal{L}}{\partial k}$
- 11: $i \leftarrow i + 1$
- 12: **end while**
- 13: **end for**

For each time step, the kernel weights were iteratively updated until either the loss dropped below a specified threshold ($\epsilon = 0.001$), or $I_{max} = 10$ iterations was reached. This local convergence criterion ensures the optimization remains lightweight and suitable for embedded implementation. The weighting coefficients were selected as $p_1 = p_2 = 1$, resulting in a symmetric quadratic penalty on the transverse angular velocity errors. The learning rate was set to 0.01. Initial angular velocities were set to -0.25 rad/s for each axis. The analytical gradients were derived by using

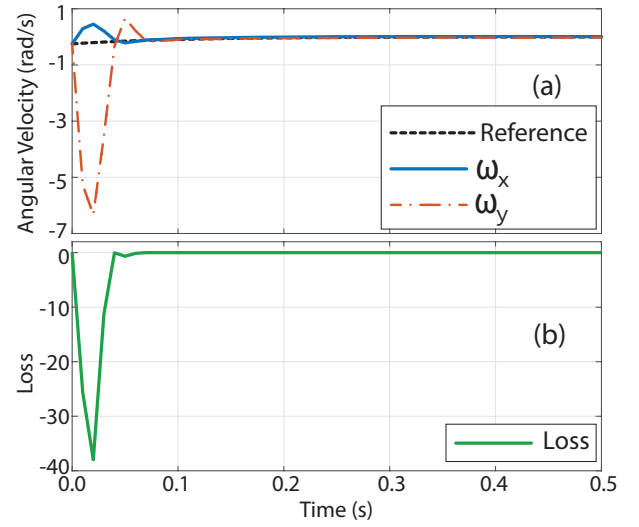


Figure 5. Case Study #3: Optimal angular velocity control with analytical gradients. a) Angular velocity response variations. b) Loss function variation.

the closed form solution, in which the Taylor series was truncated after the 15th order term. This order was selected to balance computational efficiency and approximation accuracy, based on observed trade-offs during implementation. The optimal angular velocity regularization results may be viewed in Fig. 5.

As shown in Fig. 5(a), the angular velocity profiles rapidly converge to their reference values in under 0.1 s, strongly demonstrating the control authority and effectiveness of the proposed approach. Likewise, the loss function approaches zero shortly after execution; see Fig. 5(b). It is observed that the y-axis angular velocity exhibits a noticeable overshoot. Although the cost function penalizes the transverse angular velocity errors symmetrically, nonlinear gyroscopic coupling and the structure of the gradient-based optimal control law can induce asymmetric transient responses, resulting in different overshoot magnitudes along the two axes. In this context, distinct weighting coefficients may be introduced to independently penalize angular velocity errors along different axes.

Despite the severely limited computational resources of the 8-bit microcontroller in Arduino Uno, the proposed method enabled real-time optimal control by leveraging the explicit gradient computation. With the use of the -Ofast compiler optimization flag, each control update, including optimization, was completed in approximately 27 milliseconds, demonstrating a feasible update rate. Such performance is typically unattainable on an Arduino Uno using conventional numerical optimization routines thereby showcasing the unique advantage of the proposed approach for embedded optimal control in constrained hardware environments.

To further demonstrate the computational efficiency enabled by the proposed closed-form analytical solution, the same optimal control routine was executed in MATLAB R2024b, under the computational environment defined in Case Study #1. Two methods were compared: (i) using analytically derived gradients and (ii) using MATLAB's standard fmincon solver with the fastest available configuration

(active-set). Both methods were configured with identical tolerances (10^{-3}), and maximum iteration limits (40), and no lower or upper bounds were imposed.

The optimization routine completed in 0.069 s using analytical gradients, compared to 1.3704 s with `fmincon`, across 150 optimization steps. These results highlight that the use of analytical gradients, which is made possible by the proposed closed-form solution to Euler's equations, yields nearly a 20-fold speed-up under identical computational conditions.

Case Study #4: Benchmarking with Tsiotras and Longuski's Solution

As discussed in the first section, Tsiotras and Longuski presented a seminal contribution in which they derived an analytical solution for time-varying body-fixed torques applied about all three principal axes. In their formulation, each body-fixed torque component is independently expressed as a power series expansion. In contrast, the present work applies a single power series expansion to the general zero-state solution, which may reduce the accumulation of errors associated with separate component-wise expansions.

To illustrate this advantage, a benchmarking study was performed using the conditions and torque profiles defined in Case Study #2. For both analytical approaches, the truncation order was fixed at $n_{\max} = 23$. Fig. 6 depicts the proposed solution (solid blue) alongside the solution of Tsiotras and Longuski (dashed/dotted green)⁵⁰, with a numerical integration as the reference solution (dashed red). As shown, the solution of Tsiotras and Longuski begins to diverge at approximately $t = 4.5$, whereas the proposed solution remains stable until approximately $t = 9$. This result demonstrates improved agreement with the numerical ground truth over a significantly longer time horizon under identical conditions.

Conclusion

This article presented a new closed-form analytical solution for the rotational dynamics of axisymmetric rigid bodies under time-varying torques. In contrast to existing state-of-the-art method that separately expands each body-fixed moment, the proposed approach applies a single Taylor series expansion directly to the system's zero-state response. This is enabled by a recursive formulation that efficiently computes high-order derivatives, allowing scalable, offline computation of Taylor coefficients to arbitrary accuracy. The result is a compact formulation with a single global approximation error, avoiding the accumulation of multiple truncation errors that may arise in component-wise approaches.

The closed-form structure supports flexible torque parameterizations (e.g., Gaussian kernels, Fourier series) and enables direct computation of analytical gradients. These properties make the method particularly well-suited for embedded optimal control and machine learning applications, where computational resources are limited. Simulation results demonstrated the method's practicality, achieving execution speeds nearly 20 times faster than standard numerical solvers on a severely resource-constrained platform.

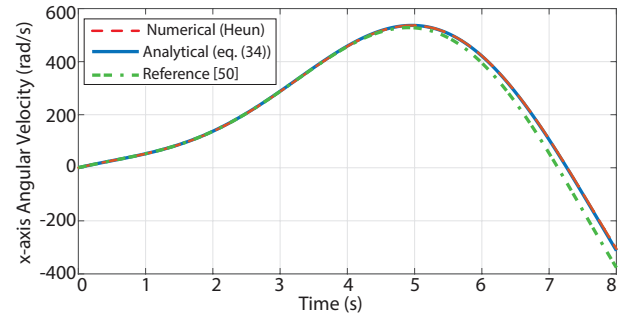


Figure 6. Case Study #4: The proposed solution is compared with Tsiotras and Longuski's solution⁵⁰. The comparison is only performed for the x-axis but a similar outcome was observed for the y-axis.

Additionally, an embedded optimal control task was solved in real time using the proposed formulation and precomputed analytical gradients on a low-cost Arduino Uno board.

As the solution is based on Taylor's series expansion, the inevitable truncation errors could cause deviations from the actual solution. Furthermore, the method is primarily applicable to axisymmetric rigid bodies and may exhibit reduced accuracy when applied to bodies lacking this property. To this end, robust adaptive and fuzzy-based methods could be integrated at a higher control level to compensate for approximation errors, thereby extending the practical operating envelope of the analytical solution^{57–61}. Investigating such hybrid analytical–adaptive frameworks constitutes a natural direction for future work. Moreover, future work will explore coupling the proposed analytical formulation with onboard learning or adaptive update mechanisms to enable real-time torque shaping and error correction on low-cost aerial platforms with limited computational resources. Specific symmetry cases⁶² will also be examined in the context of optimal control for aerial vehicles.

Acknowledgements

The author thanks P. Sendur and A. Basol.

Appendix: Binomial Expansion of $\frac{d^{n+1}x_{zs}(t)}{dt^{n+1}}$

To construct the Taylor expansion of the zero-state solution $x_{zs}(t)$, one must differentiate it repeatedly with respect to time up to the desired truncation order. The eq. (29) demonstrates the recursive structure of these derivatives. Leveraging this property, the first five time derivatives could be obtained in a subsequent fashion:

$$\begin{aligned} \frac{d^2 x_{zs}(t)}{dt^2} &= \left(\frac{\partial^2 \beta_s}{\partial t^2} S x_{zs}(t) + \frac{\partial \beta_s}{\partial t} S \frac{dx_{zs}(t)}{dt} \right) \\ &+ B \frac{du(t)}{dt} \end{aligned} \quad (47)$$

$$\begin{aligned} \frac{d^3 x_{zs}(t)}{dt^3} &= \left(\frac{\partial^3 \beta_s}{\partial t^3} S x_{zs}(t) + 2 \frac{\partial^2 \beta_s}{\partial t^2} S \frac{dx_{zs}(t)}{dt} \right. \\ &+ \left. \frac{\partial \beta_s}{\partial t} S \frac{d^2 x_{zs}(t)}{dt^2} \right) + B \frac{d^2 u(t)}{dt^2} \end{aligned} \quad (48)$$

$$\begin{aligned} \frac{d^4 x_{zs}(t)}{dt^4} &= \left(\frac{\partial^4 \beta_s}{\partial t^4} S x_{zs}(t) + 3 \frac{\partial^3 \beta_s}{\partial t^3} S \frac{dx_{zs}(t)}{dt} \right. \\ &+ 3 \frac{\partial^2 \beta_s}{\partial t^2} S \frac{d^2 x_{zs}(t)}{dt^2} + \frac{\partial \beta_s}{\partial t} S \frac{d^3 x_{zs}(t)}{dt^3} \Big) \\ &+ B \frac{d^3 u(t)}{dt^3} \end{aligned} \quad (49)$$

$$\begin{aligned} \frac{d^5 x_{zs}(t)}{dt^5} &= \left(\frac{\partial^5 \beta_s}{\partial t^5} S x_{zs}(t) + 4 \frac{\partial^4 \beta_s}{\partial t^4} S \frac{dx_{zs}(t)}{dt} \right. \\ &+ 6 \frac{\partial^3 \beta_s}{\partial t^3} S \frac{d^2 x_{zs}(t)}{dt^2} + 4 \frac{\partial^2 \beta_s}{\partial t^2} S \frac{d^3 x_{zs}(t)}{dt^3} \\ &+ \left. \frac{\partial \beta_s}{\partial t} S \frac{d^4 x_{zs}(t)}{dt^4} \right) + B \frac{d^4 u(t)}{dt^4} \end{aligned} \quad (50)$$

Note that the first derivative was presented in (29). Indeed, repeated differentiation of $x_{zs}(t)$ follows Leibniz's rule for higher-order derivatives. All higher-order derivatives can therefore be computed recursively, using previously computed derivatives. This approach is computationally efficient, and its generalization to an arbitrary $(n + 1)^{\text{th}}$ time derivative is given in (30).

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